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Diagrammatic bounds on the lace-expansion coefficients for oriented percolation

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In this note, we provide a complete proof of [2, Proposition 3.3]. For notational convenience, we use bold letters to denote vertices in $\mathbb{Z}^{d+1}$, e.g., $\boldsymbol{o} \equiv (o,0)$ and $\boldsymbol{x}$; if necessary, we denote the spatial and temporal components of a given vertex $\boldsymbol{v}$ by $\sigma_{\boldsymbol{v}}$ and $\tau_{\boldsymbol{v}}$ respectively: $\boldsymbol{v}=(\sigma_{\boldsymbol{v}},\tau_{\boldsymbol{v}})$. To identify the starting and terminal points, we write, e.g., $\varphi_p(\boldsymbol{v};\boldsymbol{x})=\mathbb{P}_p(\boldsymbol{v}\to\boldsymbol{x})$ and abbreviate it to $\varphi_p(\boldsymbol{x})$ if $\boldsymbol{v}=\boldsymbol{o}$; in particular, $\varphi_p(\boldsymbol{v};\boldsymbol{x})=\varphi_p(\boldsymbol{x}-\boldsymbol{v})$ if the model is translation-invariant. Let $\texttt{piv}(\boldsymbol{v},\boldsymbol{x})$ denote the (random) set of pivotal bonds for $\{\boldsymbol{v}\to\boldsymbol{x}\}$.

1 Bounds in terms of two-point functions

In this section, we prove bounds on $\pi_p^{(N)}(\boldsymbol{x})$ and $\Pi_p^{(N)}(\boldsymbol{x})$, for fixed $\boldsymbol{x}$, in terms of two-point functions. To prove these bounds, we do not have to assume translation-invariance.

Recall that the lace-expansion coefficients $\pi_p^{(N)}(\boldsymbol{x})$ and $\Pi_p^{(N)}(\boldsymbol{x})$ for $N\ge 1$ are defined in terms of the event

$$\tilde{E}^{(N)}_{\vec{b}_N}(\boldsymbol{x})=\{\boldsymbol{o}\rightrightarrows \underline{b}_1\}\cap\bigcap_{i=1}^{N}E\big(b_i,\underline{b}_{i+1};\tilde{\mathcal{C}}^{b_i}(\overline{b}_{i-1})\big), \tag{1.1}$$

where $\vec{b}_N=(b_1,\dots,b_N)$ is an ordered set of bonds and

$$E(b,\boldsymbol{x};\mathcal{C})=\{b\to\boldsymbol{x}\in\mathcal{C}\}\cap\big\{\nexists\, b'\in\texttt{piv}(\overline{b},\boldsymbol{x}) \text{ satisfying } \underline{b}'\in\mathcal{C}\big\}, \tag{1.2}$$

$$\tilde{\mathcal{C}}^b(\boldsymbol{v})=\{\boldsymbol{x}\in\mathbb{Z}^{d+1}:\boldsymbol{v}\to\boldsymbol{x} \text{ without using } b\}. \tag{1.3}$$

Lemma 1.

$$\pi_p^{(0)}(\boldsymbol{x})\equiv\mathbb{P}_p(\boldsymbol{o}\rightrightarrows\boldsymbol{x})\le\delta_{\boldsymbol{x},\boldsymbol{o}}+(q_p*\varphi_p)(\boldsymbol{x})^2, \tag{1.4}$$

and, for $N\ge 1$,

$$\pi_p^{(N)}(\boldsymbol{x})\equiv\sum_{\vec{b}_N}\mathbb{P}_p\big(\tilde{E}^{(N)}_{\vec{b}_N}(\boldsymbol{x})\big)\le\sum_{\substack{\boldsymbol{u}_1,\dots,\boldsymbol{u}_{N+1}\\ \boldsymbol{v}_1,\dots,\boldsymbol{v}_{N+1}\\ (\boldsymbol{u}_{N+1}=\boldsymbol{v}_{N+1}=\boldsymbol{x})}}\varphi_p(\boldsymbol{u}_1)\,\varphi_p(\boldsymbol{u}_1;\boldsymbol{v}_1)\,\varphi_p(\boldsymbol{v}_1)\prod_{i=1}^{N}\Xi_p(\boldsymbol{u}_i,\boldsymbol{v}_i;\boldsymbol{u}_{i+1},\boldsymbol{v}_{i+1}), \tag{1.5}$$

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where

$$\Xi_p(\boldsymbol{u},\boldsymbol{v};\boldsymbol{u}',\boldsymbol{v}') = \big(\xi_p^{||}(\boldsymbol{u},\boldsymbol{v};\boldsymbol{u}',\boldsymbol{v}') + \xi_p^{\times}(\boldsymbol{u},\boldsymbol{v};\boldsymbol{u}',\boldsymbol{v}')\big)\varphi_p(\boldsymbol{u}';\boldsymbol{v}')/2^{\delta_{\boldsymbol{u}',\boldsymbol{v}'}}, \tag{1.6}$$

$$\begin{cases}\xi_p^{||}(\boldsymbol{u},\boldsymbol{v};\boldsymbol{u}',\boldsymbol{v}') = (q_p * \varphi_p)(\boldsymbol{u};\boldsymbol{u}')\,(q_p*\varphi_p)(\boldsymbol{v};\boldsymbol{v}'),\\ \xi_p^{\times}(\boldsymbol{u},\boldsymbol{v};\boldsymbol{u}',\boldsymbol{v}') = (q_p * \varphi_p)(\boldsymbol{u};\boldsymbol{v}')\,(q_p*\varphi_p)(\boldsymbol{v};\boldsymbol{u}').\end{cases} \tag{1.7}$$

Proof. Since (1.4) is already proved in [2, (3.18)], it remains to show (1.5). By definition, we can easily see that

$$E(b,\boldsymbol{x};\tilde{\mathcal{C}}^b(\boldsymbol{y})) \subset \{\boldsymbol{y}\to\boldsymbol{x}\}\circ\{b\to\boldsymbol{x}\}, \tag{1.8}$$

where $E_1\circ E_2$ is the event that E_1 and E_2 occur bond-disjointly (i.e., E_1 occurs on some bond set B and E_2 occurs on $B^{\rm c}$). Similarly,

$$\{\boldsymbol{o}\rightrightarrows\boldsymbol{v}\}\cap\{\boldsymbol{o}\to\boldsymbol{x}\} \subset \bigcup_{\boldsymbol{u}}\big\{\{\boldsymbol{o}\to\boldsymbol{u}\to\boldsymbol{v}\}\circ\{\boldsymbol{o}\to\boldsymbol{v}\}\circ\{\boldsymbol{u}\to\boldsymbol{x}\}\big\}, \tag{1.9}$$

$$\begin{aligned}E(b,\boldsymbol{v};\tilde{\mathcal{C}}^b(\boldsymbol{y}))\cap\{\overline{b}\to\boldsymbol{x}\} \subset \bigcup_{\boldsymbol{u}:\tau_{\boldsymbol{u}}>\tau_{\underline{b}}} \Big\{&\big\{\{\boldsymbol{y}\to\boldsymbol{u}\to\boldsymbol{v}\}\circ\{b\to\boldsymbol{v}\}\circ\{\boldsymbol{u}\to\boldsymbol{x}\}\big\}\\ &\cup\big\{\{\boldsymbol{y}\to\boldsymbol{v}\}\circ\{b\to\boldsymbol{u}\to\boldsymbol{v}\}\circ\{\boldsymbol{u}\to\boldsymbol{x}\}\big\}\Big\}.\end{aligned} \tag{1.10}$$

To prove (1.5), we use (1.8)–(1.10) and the BK inequality and pay attention to which event depends on which time interval. For example, by (1.8),

$$\tilde{E}^{(N)}_{\vec{b}_N}(\boldsymbol{x}) \subset \tilde{E}^{(N-1)}_{\vec{b}_{N-1}}(\underline{b}_N)\cap\{\overline{b}_{N-1}\to\boldsymbol{x}\}\circ\{b_N\to\boldsymbol{x}\}. \tag{1.11}$$

Since $\tilde{E}^{(N-1)}_{\vec{b}_{N-1}}(\underline{b}_N)$ depends only on bonds before time $\tau_{\underline{b}_N}$, we can use the BK inequality to obtain

$$\sum_{b_N}\mathbb{P}_p\big(\tilde{E}^{(N)}_{\vec{b}_N}(\boldsymbol{x})\big) \le \sum_{\boldsymbol{v}_N}\mathbb{P}_p\Big(\tilde{E}^{(N-1)}_{\vec{b}_{N-1}}(\boldsymbol{v}_N)\cap\{\overline{b}_{N-1}\to\boldsymbol{x}\}\Big)\,(q_p*\varphi_p)(\boldsymbol{v}_N;\boldsymbol{x}). \tag{1.12}$$

Then, by (1.10) and the BK inequality and using the Markov property, we obtain

$$\begin{aligned}&\sum_{b_{N-1}}\mathbb{P}_p\Big(\tilde{E}^{(N-1)}_{\vec{b}_{N-1}}(\boldsymbol{v}_N)\cap\{\overline{b}_{N-1}\to\boldsymbol{x}\}\Big)\\ &\le \sum_{\substack{\boldsymbol{v}_{N-1},\boldsymbol{u}_N\\(\tau_{\boldsymbol{v}_{N-1}}<\tau_{\boldsymbol{u}_N})}}\bigg(\mathbb{P}_p\Big(\tilde{E}^{(N-2)}_{\vec{b}_{N-2}}(\boldsymbol{v}_{N-1})\cap\{\overline{b}_{N-2}\to\boldsymbol{u}_N\}\Big)(q_p*\varphi_p)(\boldsymbol{v}_{N-1};\boldsymbol{v}_N)\\ &\qquad +\mathbb{P}_p\Big(\tilde{E}^{(N-2)}_{\vec{b}_{N-2}}(\boldsymbol{v}_{N-1})\cap\{\overline{b}_{N-2}\to\boldsymbol{v}_N\}\Big)(q_p*\varphi_p)(\boldsymbol{v}_{N-1};\boldsymbol{u}_N)\bigg)\varphi_p(\boldsymbol{u}_N;\boldsymbol{v}_N)\,\varphi_p(\boldsymbol{u}_N;\boldsymbol{x}).\end{aligned} \tag{1.13}$$

Since $\tau_{\boldsymbol{u}_N}\le\tau_{\boldsymbol{v}_N}<\tau_{\boldsymbol{x}}$ (due to $(q_p*\varphi_p)(\boldsymbol{v}_N;\boldsymbol{x})$ in (1.12) and $\varphi_p(\boldsymbol{u}_N;\boldsymbol{v}_N)$ in (1.13)), we can replace the last term in (1.13) by $(q_p*\varphi_p)(\boldsymbol{u}_N;\boldsymbol{x})$, using the trivial inequality

$$\varphi_p(\boldsymbol{u};\boldsymbol{x}) \le (q_p*\varphi_p)(\boldsymbol{u};\boldsymbol{x}) \qquad (\boldsymbol{u}\ne\boldsymbol{x}). \tag{1.14}$$

Summarizing these bounds, we have

$$\sum_{b_{N-1},b_N}\mathbb{P}_p\big(\tilde{E}^{(N)}_{\vec{b}_N}(\boldsymbol{x})\big)\leq\sum_{\substack{\boldsymbol{v}_{N-1},\boldsymbol{u}_N,\boldsymbol{v}_N\\(\tau_{\boldsymbol{v}_{N-1}}<\tau_{\boldsymbol{u}_N})}}\bigg(\mathbb{P}_p\Big(\tilde{E}^{(N-2)}_{\vec{b}_{N-2}}(\boldsymbol{v}_{N-1})\cap\{\overline{b}_{N-2}\to\boldsymbol{u}_N\}\Big)(q_p*\varphi_p)(\boldsymbol{v}_{N-1};\boldsymbol{v}_N)$$
$$+\mathbb{P}_p\Big(\tilde{E}^{(N-2)}_{\vec{b}_{N-2}}(\boldsymbol{v}_{N-1})\cap\{\overline{b}_{N-2}\to\boldsymbol{v}_N\}\Big)(q_p*\varphi_p)(\boldsymbol{v}_{N-1};\boldsymbol{u}_N)\bigg)$$
$$\times\,\varphi_p(\boldsymbol{u}_N;\boldsymbol{v}_N)\,\Xi_p(\boldsymbol{u}_N,\boldsymbol{v}_N;\boldsymbol{x},\boldsymbol{x}).\tag{1.15}$$

Using (1.13)–(1.14) again, but with different variables, we obtain

$$\sum_{b_{N-2},b_{N-1},b_N}\mathbb{P}_p\big(\tilde{E}^{(N)}_{\vec{b}_N}(\boldsymbol{x})\big)\leq\sum_{\substack{\boldsymbol{u}_{N-1},\boldsymbol{u}_N\\\boldsymbol{v}_{N-2},\boldsymbol{v}_{N-1},\boldsymbol{v}_N\\(\tau_{\boldsymbol{v}_{N-2}}<\tau_{\boldsymbol{u}_{N-1}})}}\bigg(\mathbb{P}_p\Big(\tilde{E}^{(N-3)}_{\vec{b}_{N-3}}(\boldsymbol{v}_{N-2})\cap\{\overline{b}_{N-3}\to\boldsymbol{u}_{N-1}\}\Big)(q_p*\varphi_p)(\boldsymbol{v}_{N-2};\boldsymbol{v}_{N-1})$$
$$+\mathbb{P}_p\Big(\tilde{E}^{(N-3)}_{\vec{b}_{N-3}}(\boldsymbol{v}_{N-2})\cap\{\overline{b}_{N-3}\to\boldsymbol{v}_{N-1}\}\Big)(q_p*\varphi_p)(\boldsymbol{v}_{N-2};\boldsymbol{u}_{N-1})\bigg)$$
$$\times\,\varphi_p(\boldsymbol{u}_{N-1};\boldsymbol{v}_{N-1})\,\Xi_p(\boldsymbol{u}_{N-1},\boldsymbol{v}_{N-1};\boldsymbol{u}_N,\boldsymbol{v}_N)\,\Xi_p(\boldsymbol{u}_N,\boldsymbol{v}_N;\boldsymbol{x},\boldsymbol{x}).\tag{1.16}$$

We repeat this procedure until we arrive at

$$\sum_{\vec{b}_N}\mathbb{P}_p\big(\tilde{E}^{(N)}_{\vec{b}_N}(\boldsymbol{x})\big)\leq\sum_{\substack{\boldsymbol{u}_2,\dots,\boldsymbol{u}_N\\\boldsymbol{v}_1,\dots,\boldsymbol{v}_N\\(\tau_{\boldsymbol{v}_1}<\tau_{\boldsymbol{u}_2})}}\bigg(\mathbb{P}_p\Big(\{\boldsymbol{o}\rightrightarrows\boldsymbol{v}_1\}\cap\{\boldsymbol{o}\to\boldsymbol{u}_2\}\Big)(q_p*\varphi_p)(\boldsymbol{v}_1;\boldsymbol{v}_2)$$
$$+\mathbb{P}_p\Big(\{\boldsymbol{o}\rightrightarrows\boldsymbol{v}_1\}\cap\{\boldsymbol{o}\to\boldsymbol{v}_2\}\Big)(q_p*\varphi_p)(\boldsymbol{v}_1;\boldsymbol{u}_2)\bigg)\varphi_p(\boldsymbol{u}_2;\boldsymbol{v}_2)$$
$$\times\prod_{i=2}^{N-1}\Xi_p(\boldsymbol{u}_i,\boldsymbol{v}_i;\boldsymbol{u}_{i+1},\boldsymbol{v}_{i+1})\,\Xi_p(\boldsymbol{u}_N,\boldsymbol{v}_N;\boldsymbol{x},\boldsymbol{x}).\tag{1.17}$$

By (1.9) and the BK inequality and using the Markov property and (1.14) under the restriction $\tau_{\boldsymbol{v}_1}<\tau_{\boldsymbol{u}_2}$, we obtain (1.5). ■

Lemma 2. *For $N\geq 1$,*

$$\Pi^{(N)}_p(\boldsymbol{x})\equiv\sum_{\vec{b}_N,b}\sum_{j=1}^{N}\mathbb{P}_p\Big(\tilde{E}^{(N)}_{\vec{b}_N}(\boldsymbol{x})\cap\big\{b=b_j\ \textit{or}\ b\in\texttt{piv}(\overline{b}_j,\underline{b}_{j+1})\big\}\Big)$$
$$\leq\sum_{\substack{\boldsymbol{u}_1,\dots,\boldsymbol{u}_{N+1}\\\boldsymbol{v}_1,\dots,\boldsymbol{v}_{N+1}\\(\boldsymbol{u}_{N+1}=\boldsymbol{v}_{N+1}=\boldsymbol{x})}}\varphi_p(\boldsymbol{u}_1)\,\varphi_p(\boldsymbol{u}_1;\boldsymbol{v}_1)\,\varphi_p(\boldsymbol{v}_1)\sum_{j=1}^{N}\prod_{i\neq j}\Xi_p(\boldsymbol{u}_i,\boldsymbol{v}_i;\boldsymbol{u}_{i+1},\boldsymbol{v}_{i+1})\tag{1.18}$$
$$\times\Big(\Xi_p(\boldsymbol{u}_j,\boldsymbol{v}_j;\boldsymbol{u}_{j+1},\boldsymbol{v}_{j+1})+\Theta_p(\boldsymbol{u}_j,\boldsymbol{v}_j;\boldsymbol{u}_{j+1},\boldsymbol{v}_{j+1})+\Theta'_p(\boldsymbol{u}_j,\boldsymbol{v}_j;\boldsymbol{u}_{j+1},\boldsymbol{v}_{j+1})\Big),$$

where the empty product $\prod_{i\neq j}\Xi_p(\boldsymbol{u}_i,\boldsymbol{v}_i;\boldsymbol{u}_{i+1},\boldsymbol{v}_{i+1})$ *for the case of* $N=1$ *is 1 by convention, and*

$$\Theta_p(\boldsymbol{u},\boldsymbol{v};\boldsymbol{u}',\boldsymbol{v}')=\big(\theta_p^{||}(\boldsymbol{u},\boldsymbol{v};\boldsymbol{u}',\boldsymbol{v}')+\theta_p^{\times}(\boldsymbol{u},\boldsymbol{v};\boldsymbol{u}',\boldsymbol{v}')\big)\varphi_p(\boldsymbol{u}';\boldsymbol{v}')/2^{\delta_{\boldsymbol{u}',\boldsymbol{v}'}},\tag{1.19}$$

$$\begin{cases}\theta_p^{||}(\boldsymbol{u},\boldsymbol{v};\boldsymbol{u}',\boldsymbol{v}')=(q_p*\varphi_p)(\boldsymbol{u};\boldsymbol{u}')\,(q_p*\varphi_p*q_p*\varphi_p)(\boldsymbol{v};\boldsymbol{v}'),\\ \theta_p^{\times}(\boldsymbol{u},\boldsymbol{v};\boldsymbol{u}',\boldsymbol{v}')=(q_p*\varphi_p)(\boldsymbol{u};\boldsymbol{v}')\,(q_p*\varphi_p*q_p*\varphi_p)(\boldsymbol{v};\boldsymbol{u}'),\end{cases}\tag{1.20}$$

$$\Theta'_p(\boldsymbol{u},\boldsymbol{v};\boldsymbol{u}',\boldsymbol{v}')=(q_p*\varphi_p)(\boldsymbol{u};\boldsymbol{v}')\,(q_p*\varphi_p)(\boldsymbol{v};\boldsymbol{u}')\,(\varphi_p*q_p*\varphi_p)(\boldsymbol{u}';\boldsymbol{v}').\tag{1.21}$$

Proof. Since

$$\sum_{\vec b_N,b}\mathbb{P}_p\Big(\tilde E^{(N)}_{\vec b_N}(\boldsymbol{x})\cap\big\{b=b_j\text{ or }b\in\texttt{piv}(\overline{b}_j,\underline{b}_{j+1})\big\}\Big)=\pi_p^{(N)}(\boldsymbol{x})+\sum_{\vec b_N,b}\mathbb{P}_p\Big(\tilde E^{(N)}_{\vec b_N}(\boldsymbol{x})\cap\big\{b\in\texttt{piv}(\overline{b}_j,\underline{b}_{j+1})\big\}\Big),\tag{1.22}$$

it suffices to investigate the sum on the right-hand side. To do so, we use the following relations that are similar to (1.8) and (1.10):

$$E(b',\boldsymbol{x};\tilde{\mathcal{C}}^{b'}(\boldsymbol{y}))\cap\big\{b\in\texttt{piv}(\overline{b}',\boldsymbol{x})\big\}\subset\{\boldsymbol{y}\to\boldsymbol{x}\}\circ\{b'\to b\to\boldsymbol{x}\},\tag{1.23}$$

$$\begin{aligned}E(b',\boldsymbol{v};\tilde{\mathcal{C}}^{b'}(\boldsymbol{y}))\cap\big\{b\in\texttt{piv}(\overline{b}',\boldsymbol{v})\big\}\cap\{\overline{b}'\to\boldsymbol{x}\}\subset\bigcup_{\boldsymbol{u}:\tau_{\boldsymbol{u}}>\tau_{\underline{b}'}}&\Big\{\big\{\{\boldsymbol{y}\to\boldsymbol{u}\to\boldsymbol{v}\}\circ\{b'\to b\to\boldsymbol{v}\}\circ\{\boldsymbol{u}\to\boldsymbol{x}\}\big\}\\&\cup\big\{\{\boldsymbol{y}\to\boldsymbol{v}\}\circ\{b'\to b\to\boldsymbol{u}\to\boldsymbol{v}\}\circ\{\boldsymbol{u}\to\boldsymbol{x}\}\big\}\\&\cup\big\{\{\boldsymbol{y}\to\boldsymbol{v}\}\circ\{b'\to\boldsymbol{u}\to b\to\boldsymbol{v}\}\circ\{\boldsymbol{u}\to\boldsymbol{x}\}\big\}\Big\}.\end{aligned}\tag{1.24}$$

First we let $j=N$. By (1.23) and using the BK inequality and the Markov property, we obtain

$$\sum_{b_N,b}\mathbb{P}_p\Big(\tilde E^{(N)}_{\vec b_N}(\boldsymbol{x})\cap\big\{b\in\texttt{piv}(\overline{b}_N,\boldsymbol{x})\big\}\Big)\le\sum_{\boldsymbol{v}_N}\mathbb{P}_p\Big(\tilde E^{(N-1)}_{\vec b_{N-1}}(\boldsymbol{v}_N)\cap\{\overline{b}_{N-1}\to\boldsymbol{x}\}\Big)(q_p*\varphi_p*q_p*\varphi_p)(\boldsymbol{v}_N;\boldsymbol{x}),\tag{1.25}$$

which is equivalent to (1.12), except for the last term $(q_p*\varphi_p*q_p*\varphi_p)(\boldsymbol{v}_N;\boldsymbol{x})$. Therefore, by following the same line as in (1.13)–(1.17), we obtain

$$\begin{aligned}\sum_{\vec b_N,b}\mathbb{P}_p\Big(\tilde E^{(N)}_{\vec b_N}(\boldsymbol{x})\cap\big\{b\in\texttt{piv}(\overline{b}_N,\boldsymbol{x})\big\}\Big)\le\sum_{\substack{\boldsymbol{u}_1,\dots,\boldsymbol{u}_N\\ \boldsymbol{v}_1,\dots,\boldsymbol{v}_N}}\varphi_p(\boldsymbol{u}_1)\,\varphi_p(\boldsymbol{u}_1;\boldsymbol{v}_1)\,\varphi_p(\boldsymbol{v}_1)\prod_{i=1}^{N-1}\Xi_p(\boldsymbol{u}_i,\boldsymbol{v}_i;\boldsymbol{u}_{i+1},\boldsymbol{v}_{i+1})&\\ \times\Theta_p(\boldsymbol{u}_N,\boldsymbol{v}_N;\boldsymbol{x},\boldsymbol{x}).&\end{aligned}\tag{1.26}$$

Applying (1.5) to $\pi_p^{(N)}(\boldsymbol{x})$ in (1.22) and using $\Theta'_p(\boldsymbol{u},\boldsymbol{v};\boldsymbol{u}',\boldsymbol{v}')=0$ for $\boldsymbol{u}'=\boldsymbol{v}'$ (since $(\varphi_p*q_p*\varphi_p)(\boldsymbol{u}';\boldsymbol{v}')$ in (1.21) is zero if $\boldsymbol{u}'=\boldsymbol{v}'$), we obtain the term for $j=N$ in (1.18).

Next we let $j<N$. Following the same line as in (1.12)–(1.16), we obtain

$$\begin{aligned}
&\sum_{\vec b_N,b}\mathbb{P}_p\Big(\tilde E^{(N)}_{\vec b_N}(\boldsymbol{x})\cap\big\{b\in\mathtt{piv}(\overline{b}_j,\underline{b}_{j+1})\big\}\Big)\\
&\le\sum_{\substack{\boldsymbol{u}_{j+2},\dots,\boldsymbol{u}_N\\ \boldsymbol{v}_{j+1},\dots,\boldsymbol{v}_N\\ (\tau_{\boldsymbol{v}_{j+1}}<\tau_{\boldsymbol{u}_{j+2}})}}\sum_{\vec b_j,b}\bigg(\mathbb{P}_p\Big(\tilde E^{(j)}_{\vec b_j}(\boldsymbol{v}_{j+1})\cap\big\{b\in\mathtt{piv}(\overline{b}_j,\boldsymbol{v}_{j+1})\big\}\cap\{\overline{b}_j\to\boldsymbol{u}_{j+2}\}\Big)(q_p*\varphi_p)(\boldsymbol{v}_{j+1};\boldsymbol{v}_{j+2})\\
&\qquad+\mathbb{P}_p\Big(\tilde E^{(j)}_{\vec b_j}(\boldsymbol{v}_{j+1})\cap\big\{b\in\mathtt{piv}(\overline{b}_j,\boldsymbol{v}_{j+1})\big\}\cap\{\overline{b}_j\to\boldsymbol{v}_{j+2}\}\Big)(q_p*\varphi_p)(\boldsymbol{v}_{j+1};\boldsymbol{u}_{j+2})\bigg)\\
&\qquad\times\varphi_p(\boldsymbol{u}_{j+2};\boldsymbol{v}_{j+2})\prod_{i=j+2}^{N-1}\Xi_p(\boldsymbol{u}_i,\boldsymbol{v}_i;\boldsymbol{u}_{i+1},\boldsymbol{v}_{i+1})\,\Xi_p(\boldsymbol{u}_N,\boldsymbol{v}_N;\boldsymbol{x},\boldsymbol{x}).
\end{aligned}\tag{1.27}$$

Then, by (1.24) and using the BK inequality and the Markov property,

$$\begin{aligned}
&\sum_{b_j,b}\mathbb{P}_p\Big(\tilde E^{(j)}_{\vec b_j}(\boldsymbol{v}_{j+1})\cap\big\{b\in\mathtt{piv}(\overline{b}_j,\boldsymbol{v}_{j+1})\big\}\cap\{\overline{b}_j\to\boldsymbol{u}_{j+2}\}\Big)\\
&\le\sum_{\substack{\boldsymbol{v}_j,\boldsymbol{u}_{j+1}\\ (\tau_{\boldsymbol{v}_j}<\tau_{\boldsymbol{u}_{j+1}})}}\Bigg(\bigg(\mathbb{P}_p\Big(\tilde E^{(j-1)}_{\vec b_{j-1}}(\boldsymbol{v}_j)\cap\{\overline{b}_{j-1}\to\boldsymbol{u}_{j+1}\}\Big)(q_p*\varphi_p*q_p*\varphi_p)(\boldsymbol{v}_j;\boldsymbol{v}_{j+1})\\
&\qquad+\mathbb{P}_p\Big(\tilde E^{(j-1)}_{\vec b_{j-1}}(\boldsymbol{v}_j)\cap\{\overline{b}_{j-1}\to\boldsymbol{v}_{j+1}\}\Big)(q_p*\varphi_p*q_p*\varphi_p)(\boldsymbol{v}_j;\boldsymbol{u}_{j+1})\bigg)\varphi_p(\boldsymbol{u}_{j+1};\boldsymbol{v}_{j+1})\\
&\qquad+\mathbb{P}_p\Big(\tilde E^{(j-1)}_{\vec b_{j-1}}(\boldsymbol{v}_j)\cap\{\overline{b}_{j-1}\to\boldsymbol{v}_{j+1}\}\Big)(q_p*\varphi_p)(\boldsymbol{v}_j;\boldsymbol{u}_{j+1})\,(\varphi_p*q_p*\varphi_p)(\boldsymbol{u}_{j+1};\boldsymbol{v}_{j+1})\Bigg)\varphi_p(\boldsymbol{u}_{j+1};\boldsymbol{u}_{j+2}),
\end{aligned}\tag{1.28}$$

where the last term can be replaced by $(q_p*\varphi_p)(\boldsymbol{u}_{j+1};\boldsymbol{u}_{j+2})$, because $\tau_{\boldsymbol{u}_{j+1}}\le\tau_{\boldsymbol{v}_{j+1}}<\tau_{\boldsymbol{u}_{j+2}}$ (due to the restriction in (1.27) and the factors $\varphi_p(\boldsymbol{u}_{j+1};\boldsymbol{v}_{j+1})$ and $(\varphi_p*q_p*\varphi_p)(\boldsymbol{u}_{j+1};\boldsymbol{v}_{j+1})$ in (1.28)). Using (1.28) as well as that with $\boldsymbol{u}_{j+2}$ replaced by $\boldsymbol{v}_{j+2}$, we obtain

$$\begin{aligned}
(1.27)\le&\sum_{\substack{\boldsymbol{u}_{j+1},\dots,\boldsymbol{u}_N\\ \boldsymbol{v}_j,\dots,\boldsymbol{v}_N\\ (\tau_{\boldsymbol{v}_j}<\tau_{\boldsymbol{u}_{j+1}})}}\sum_{\vec b_{j-1}}\Bigg(\bigg(\mathbb{P}_p\Big(\tilde E^{(j-1)}_{\vec b_{j-1}}(\boldsymbol{v}_j)\cap\{\overline{b}_{j-1}\to\boldsymbol{u}_{j+1}\}\Big)(q_p*\varphi_p*q_p*\varphi_p)(\boldsymbol{v}_j;\boldsymbol{v}_{j+1})\\
&\qquad+\mathbb{P}_p\Big(\tilde E^{(j-1)}_{\vec b_{j-1}}(\boldsymbol{v}_j)\cap\{\overline{b}_{j-1}\to\boldsymbol{v}_{j+1}\}\Big)\,(q_p*\varphi_p*q_p*\varphi_p)(\boldsymbol{v}_j;\boldsymbol{u}_{j+1})\bigg)\varphi_p(\boldsymbol{u}_{j+1};\boldsymbol{v}_{j+1})\\
&\qquad+\mathbb{P}_p\Big(\tilde E^{(j-1)}_{\vec b_{j-1}}(\boldsymbol{v}_j)\cap\{\overline{b}_{j-1}\to\boldsymbol{v}_{j+1}\}\Big)\,(q_p*\varphi_p)(\boldsymbol{v}_j;\boldsymbol{u}_{j+1})\,(\varphi_p*q_p*\varphi_p)(\boldsymbol{u}_{j+1};\boldsymbol{v}_{j+1})\Bigg)\\
&\qquad\times\prod_{i=j+1}^{N-1}\Xi_p(\boldsymbol{u}_i,\boldsymbol{v}_i;\boldsymbol{u}_{i+1},\boldsymbol{v}_{i+1})\,\Xi_p(\boldsymbol{u}_N,\boldsymbol{v}_N;\boldsymbol{x},\boldsymbol{x}).
\end{aligned}\tag{1.29}$$

Repeatedly using (1.13)–(1.14) with different variables, we finally arrive at

$$
\begin{aligned}
(1.29) \le \sum_{\substack{\boldsymbol{u}_j,\dots,\boldsymbol{u}_{N+1} \\ \boldsymbol{v}_{j-1},\dots,\boldsymbol{v}_{N+1} \\ (\boldsymbol{u}_{N+1}=\boldsymbol{v}_{N+1}=\boldsymbol{x}) \\ (\tau_{\boldsymbol{v}_{j-1}}<\tau_{\boldsymbol{u}_j})}} \sum_{\vec{b}_{j-2}} \bigg(& \mathbb{P}_p\Big(\tilde{E}^{(j-2)}_{\vec{b}_{j-2}}(\boldsymbol{v}_{j-1}) \cap \{\overline{b}_{j-2} \to \boldsymbol{u}_j\}\Big)(q_p * \varphi_p * q_p * \varphi_p)(\boldsymbol{v}_{j-1};\boldsymbol{v}_j) \\
& + \mathbb{P}_p\Big(\tilde{E}^{(j-2)}_{\vec{b}_{j-2}}(\boldsymbol{v}_{j-1}) \cap \{\overline{b}_{j-2} \to \boldsymbol{v}_j\}\Big)\,(q_p * \varphi_p * q_p * \varphi_p)(\boldsymbol{v}_{j-1};\boldsymbol{u}_j)\bigg)\varphi_p(\boldsymbol{u}_j;\boldsymbol{v}_j) \\
& \times \Big(\Theta_p(\boldsymbol{u}_j,\boldsymbol{v}_j;\boldsymbol{u}_{j+1},\boldsymbol{v}_{j+1}) + \Theta'_p(\boldsymbol{u}_j,\boldsymbol{v}_j;\boldsymbol{u}_{j+1},\boldsymbol{v}_{j+1})\Big) \prod_{i=j+1}^{N} \Xi_p(\boldsymbol{u}_i,\boldsymbol{v}_i;\boldsymbol{u}_{i+1},\boldsymbol{v}_{i+1}) \\
\vdots \\
\le \sum_{\substack{\boldsymbol{u}_1,\dots,\boldsymbol{u}_{N+1} \\ \boldsymbol{v}_1,\dots,\boldsymbol{v}_{N+1} \\ (\boldsymbol{u}_{N+1}=\boldsymbol{v}_{N+1}=\boldsymbol{x})}} & \varphi_p(\boldsymbol{u}_1)\,\varphi_p(\boldsymbol{v}_1-\boldsymbol{u}_1)\,\varphi_p(\boldsymbol{v}_1)\prod_{i\ne j}\Xi_p(\boldsymbol{u}_i,\boldsymbol{v}_i;\boldsymbol{u}_{i+1},\boldsymbol{v}_{i+1}) \\
& \times \Big(\Theta_p(\boldsymbol{u}_j,\boldsymbol{v}_j;\boldsymbol{u}_{j+1},\boldsymbol{v}_{j+1}) + \Theta'_p(\boldsymbol{u}_j,\boldsymbol{v}_j;\boldsymbol{u}_{j+1},\boldsymbol{v}_{j+1})\Big). \qquad (1.30)
\end{aligned}
$$

Combining this with the bound (1.5) on $\pi_p^{(N)}(\boldsymbol{x})$ in (1.22), we obtain the term for $j < N$ in (1.18).

The proof of (1.18) is completed by summing the above bounds over $j = 1, \dots, N$. ■

2 Proof of [2, Proposition 3.3]

In this section, we prove [2, Proposition 3.3] using Lemmas 1–2 and assuming translation-invariance.

Let $\varphi_p^{(m)}(\boldsymbol{v};\boldsymbol{x}) = \varphi_p(\boldsymbol{v};\boldsymbol{x})m^{\tau_{\boldsymbol{x}}-\tau_{\boldsymbol{v}}}$. Recall that the weighted bubble $W_p^{(m)}(k)$ and the triangles $T_p^{(m)}$ and $\tilde{T}_p$ are defined as

$$
W_p^{(m)}(k) = \sup_{\boldsymbol{x}\in\mathbb{Z}^{d+1}} \sum_{\boldsymbol{v}} \big(1-\cos(k\cdot\sigma_{\boldsymbol{v}})\big) \times \begin{cases} (q_p * \varphi_p)(\boldsymbol{v})\,(mq_p * \varphi_p^{(m)})(\boldsymbol{x};\boldsymbol{v}) & (m<1), \\ (mq_p * \varphi_p^{(m)})(\boldsymbol{v})\,(q_p * \varphi_p)(\boldsymbol{x};\boldsymbol{v}) & (m\ge 1), \end{cases} \tag{2.1}
$$

$$
T_p^{(m)} = \sup_{\boldsymbol{x}\in\mathbb{Z}^{d+1}} \sum_{\boldsymbol{v}} (q_p * \varphi_p * \varphi_p)(\boldsymbol{v})\,(mq_p * \varphi_p^{(m)})(\boldsymbol{x};\boldsymbol{v}), \tag{2.2}
$$

$$
\tilde{T}_p = \sup_{\boldsymbol{x}\in\mathbb{Z}^{d+1}} \sum_{\boldsymbol{v}} (q_p * \varphi_p * q_p * \varphi_p)(\boldsymbol{v})\,(q_p * \varphi_p)(\boldsymbol{x};\boldsymbol{v}), \tag{2.3}
$$

and that the square $S_p^{(m)}$ and the H-shaped diagram H_p are defined as (cf., [2, Figure 2])

$$
S_p^{(m)} = \sup_{\boldsymbol{x}\in\mathbb{Z}^{d+1}} \sum_{\boldsymbol{v}} (q_p * \varphi_p * \varphi_p * \varphi_p)(\boldsymbol{v})\,(mq_p * \varphi_p^{(m)})(\boldsymbol{x};\boldsymbol{v}), \tag{2.4}
$$

$$
H_p = \sup_{\boldsymbol{x},\boldsymbol{y}\in\mathbb{Z}^{d+1}} \sum_{\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}} (q_p * \varphi_p)(\boldsymbol{u})\,(\varphi_p * q_p * \varphi_p)(\boldsymbol{u};\boldsymbol{v})\,(q_p * \varphi_p)(\boldsymbol{x};\boldsymbol{v})\,(q_p * \varphi_p)(\boldsymbol{u};\boldsymbol{w})\,(q_p * \varphi_p)(\boldsymbol{v};\boldsymbol{y}+\boldsymbol{w}). \tag{2.5}
$$

[2, Proposition 3.3] is an immediate consequence of the following lemma:

Lemma 3. *(i) For $N \geq 0$ and $\ell = 0, 1, 2$,*

$$\sum_{\boldsymbol{x}\in\mathbb{Z}^d\times\mathbb{N}} \tau_{\boldsymbol{x}}^{\ell}\,\pi_p^{(N)}(\boldsymbol{x})m^{\tau_{\boldsymbol{x}}} \leq (N+1)^{\ell}(1+2T_p^{(m)})(2T_p^{(m)})^{(N-1)\vee 0} \times \begin{cases} T_p^{(m)} & (\ell \leq 1), \\ S_p^{(m)} & (\ell = 2), \end{cases} \tag{2.6}$$

$$\sum_{\boldsymbol{x}\in\mathbb{Z}^d\times\mathbb{Z}_+} \big(1-\cos(k\cdot\sigma_{\boldsymbol{x}})\big)\pi_p^{(N)}(\boldsymbol{x})m^{\tau_{\boldsymbol{x}}} \leq 3(N+1)^2(1+2T_p^{(m)})(2T_p^{(m)})^{(N-1)\vee 0}W_p^{(m)}(k). \tag{2.7}$$

(ii) For $N \geq 1$,

$$\sum_{\boldsymbol{x}\in\mathbb{Z}^d\times\mathbb{Z}_+} \Pi_p^{(N)}(\boldsymbol{x}) \leq N(1+2T_p^{(1)})\Big((T_p^{(1)}+\tilde{T}_p)(2T_p^{(1)})^{N-1} + H_p(2T_p^{(1)})^{(N-2)\vee 0}\Big). \tag{2.8}$$

Proof of Lemma 3(i). First we prove (2.6)–(2.7) for $N = 0$. By (1.4), we readily obtain

$$\sum_{\boldsymbol{x}\in\mathbb{Z}^d\times\mathbb{N}} \tau_{\boldsymbol{x}}^{\ell}\,\pi_p^{(0)}(\boldsymbol{x})m^{\tau_{\boldsymbol{x}}} \leq \sum_{\boldsymbol{x}} \tau_{\boldsymbol{x}}^{\ell}\,(q_p*\varphi_p)(\boldsymbol{x})\,(mq_p*\varphi_p^{(m)})(\boldsymbol{x}) \leq \begin{cases} T_p^{(m)} & (\ell \leq 1), \\ S_p^{(m)} & (\ell = 2), \end{cases} \tag{2.9}$$

where we have used (cf., [3, (5.17)])

$$\tau_{\boldsymbol{x}}\,(q_p*\varphi_p)(\boldsymbol{x}) = \sum_{t=1}^{\tau_{\boldsymbol{x}}} (q_p*\varphi_p)(\boldsymbol{x}) \leq \sum_{t=1}^{\tau_{\boldsymbol{x}}} \sum_{\boldsymbol{v}:\tau_{\boldsymbol{v}}=t} (q_p*\varphi_p)(\boldsymbol{v})\,\varphi_p(\boldsymbol{v};\boldsymbol{x}) = (q_p*\varphi_p*\varphi_p)(\boldsymbol{x}), \tag{2.10}$$

$$\tau_{\boldsymbol{x}}^2\,(q_p*\varphi_p)(\boldsymbol{x}) \overset{\because(2.10)}{\leq} \tau_{\boldsymbol{x}}\,(q_p*\varphi_p*\varphi_p)(\boldsymbol{x}) \overset{\because(2.10)}{\leq} (q_p*\varphi_p*\varphi_p*\varphi_p)(\boldsymbol{x}). \tag{2.11}$$

We note that we have multiplied one of the two diagram lines (i.e., $(q_p*\varphi_p)(\boldsymbol{x})$) by $\tau_{\boldsymbol{x}}^{\ell}$ and the other by $m^{\tau_{\boldsymbol{x}}}$. If we multiply either $(q_p*\varphi_p)(\boldsymbol{x})$ or $(mq_p*\varphi_p^{(m)})(\boldsymbol{x})$ (depending on whether $m<1$ or $m\geq 1$) by $1-\cos(k\cdot\sigma_{\boldsymbol{x}})$ instead of $\tau_{\boldsymbol{x}}^{\ell}$, we obtain

$$\sum_{\boldsymbol{x}} \big(1-\cos(k\cdot\sigma_{\boldsymbol{x}})\big)\,\pi_p^{(0)}(\boldsymbol{x})m^{\tau_{\boldsymbol{x}}} \leq W_p^{(m)}(k), \tag{2.12}$$

as required.

Next we prove (2.6) for $N \geq 1$ and $\ell = 0$. We note that, as in the $N=0$ case above, there are two "external" diagram lines from $\boldsymbol{o}$ to $\boldsymbol{x}$ in each of the 2^{N-1} bounding diagrams in (1.5). Each line looks like

$$\varphi_p(\boldsymbol{y}_1)\prod_{i=1}^{N-1}(q_p*\varphi_p)(\boldsymbol{y}_i;\boldsymbol{y}_{i+1})\,(q_p*\varphi_p)(\boldsymbol{y}_N;\boldsymbol{x}), \tag{2.13}$$

where each $\boldsymbol{y}_i$ is either $\boldsymbol{u}_i$ or $\boldsymbol{v}_i$ in (1.5); denote the line with $\boldsymbol{y}_1 = \boldsymbol{v}_1$ by $\omega_{\boldsymbol{v}_1} \equiv (\boldsymbol{o}, \boldsymbol{v}_1, \dots, \boldsymbol{x})$ and the other by $\omega_{\boldsymbol{u}_1} \equiv (\boldsymbol{o}, \boldsymbol{u}_1, \dots, \boldsymbol{x})$. Multiplying $\omega_{\boldsymbol{v}_1}$ by $m^{\tau_{\boldsymbol{x}}}$ and using

$$\left.\begin{array}{c} 2\displaystyle\sum_{\boldsymbol{x}} \Xi_p(\boldsymbol{o},\boldsymbol{y};\boldsymbol{x},\boldsymbol{x})m^{\tau_{\boldsymbol{x}}} \\ \displaystyle\sum_{\boldsymbol{u},\boldsymbol{v}} \big(\xi_p^{||}(\boldsymbol{o},\boldsymbol{y};\boldsymbol{u},\boldsymbol{v})m^{\tau_{\boldsymbol{u}}} + \xi_p^{\times}(\boldsymbol{o},\boldsymbol{y};\boldsymbol{u},\boldsymbol{v})m^{\tau_{\boldsymbol{v}}}\big)\varphi_p(\boldsymbol{u};\boldsymbol{v}) \\ \displaystyle\sum_{\boldsymbol{u},\boldsymbol{v}} \big(\xi_p^{||}(\boldsymbol{y},\boldsymbol{o};\boldsymbol{u},\boldsymbol{v})m^{\tau_{\boldsymbol{v}}} + \xi_p^{\times}(\boldsymbol{y},\boldsymbol{o};\boldsymbol{u},\boldsymbol{v})m^{\tau_{\boldsymbol{u}}}\big)\varphi_p(\boldsymbol{u};\boldsymbol{v}) \end{array}\right\} \leq 2T_p^{(m)} \qquad (\boldsymbol{y}\in\mathbb{Z}^{d+1}), \tag{2.14}$$

and

$$\sum_{\boldsymbol{u},\boldsymbol{v}} \varphi_p(\boldsymbol{u})\, \varphi_p(\boldsymbol{u};\boldsymbol{v})\, \varphi_p(\boldsymbol{v}) m^{\tau_{\boldsymbol{v}}} \le 1 + \sum_{\boldsymbol{v}\neq\boldsymbol{o}} (\varphi_p * \varphi_p)(\boldsymbol{v})\, (mq_p * \varphi_p^{(m)})(\boldsymbol{v}) \le 1 + 2T_p^{(m)}, \tag{2.15}$$

we obtain

$$\sum_{\boldsymbol{x}} \pi_p^{(N)}(\boldsymbol{x}) m^{\tau_{\boldsymbol{x}}} \le (1+2T_p^{(m)})(2T_p^{(m)})^{N-1} T_p^{(m)} \qquad (N \ge 1), \tag{2.16}$$

as required.

Before proceeding the proof, we define (cf., (1.6))

$$\tilde{\Xi}_p(\boldsymbol{u},\boldsymbol{v};\boldsymbol{u}',\boldsymbol{v}') = \varphi_p(\boldsymbol{u};\boldsymbol{v}) \big(\xi_p^{||}(\boldsymbol{u},\boldsymbol{v};\boldsymbol{u}',\boldsymbol{v}') + \xi_p^{\times}(\boldsymbol{u},\boldsymbol{v};\boldsymbol{u}',\boldsymbol{v}')\big)/2^{\delta_{\boldsymbol{u}',\boldsymbol{v}'}}, \tag{2.17}$$

which satisfies similar bounds to (2.14), due to translation-invariance. We note that, by using (2.17), the bound in (1.5) can be reorganized as

$$\sum_{\substack{\boldsymbol{u}_1,\dots,\boldsymbol{u}_{N+1}\\ \boldsymbol{v}_1,\dots,\boldsymbol{v}_{N+1}\\ (\boldsymbol{u}_{N+1}=\boldsymbol{v}_{N+1}=\boldsymbol{x})}} \varphi_p(\boldsymbol{u}_1)\, \varphi_p(\boldsymbol{v}_1) \prod_{i=1}^{N} \tilde{\Xi}_p(\boldsymbol{u}_i,\boldsymbol{v}_i;\boldsymbol{u}_{i+1},\boldsymbol{v}_{i+1}), \tag{2.18}$$

or, for $j = 1,\dots,N$, as

$$\begin{aligned} &\sum_{\substack{\boldsymbol{u}_1,\dots,\boldsymbol{u}_{N+1}\\ \boldsymbol{v}_1,\dots,\boldsymbol{v}_{N+1}\\ (\boldsymbol{u}_{N+1}=\boldsymbol{v}_{N+1}=\boldsymbol{x})}} \varphi_p(\boldsymbol{u}_1)\, \varphi_p(\boldsymbol{u}_1;\boldsymbol{v}_1)\, \varphi_p(\boldsymbol{v}_1) \bigg(\prod_{i=1}^{j-1} \Xi_p(\boldsymbol{u}_i,\boldsymbol{v}_i;\boldsymbol{u}_{i+1},\boldsymbol{v}_{i+1}) \bigg) \\ &\qquad \times \Big(\xi_p^{||}(\boldsymbol{u}_j,\boldsymbol{v}_j;\boldsymbol{u}_{j+1},\boldsymbol{v}_{j+1}) + \xi_p^{\times}(\boldsymbol{u}_j,\boldsymbol{v}_j;\boldsymbol{u}_{j+1},\boldsymbol{v}_{j+1}) \Big) \bigg(\prod_{i=j+1}^{N} \tilde{\Xi}_p(\boldsymbol{u}_i,\boldsymbol{v}_i;\boldsymbol{u}_{i+1},\boldsymbol{v}_{i+1}) \bigg). \end{aligned} \tag{2.19}$$

Now we prove (2.6) for $N \ge 1$ and $\ell = 1, 2$. To do so, we multiply $\omega_{\boldsymbol{v}_1}$ by $m^{\tau_{\boldsymbol{x}}}$ as before, and multiply $\omega_{\boldsymbol{u}_1} = (\boldsymbol{o}, \omega_{\boldsymbol{u}_1}^{(1)}, \dots, \omega_{\boldsymbol{u}_1}^{(N+1)})$, where $\omega_{\boldsymbol{u}_1}^{(1)} = \boldsymbol{u}_1$, $\omega_{\boldsymbol{u}_1}^{(i)} \in \{\boldsymbol{u}_i, \boldsymbol{v}_i\}$ for $i = 2,\dots,N$ and $\omega_{\boldsymbol{u}_1}^{(N+1)} = \boldsymbol{x}$, by $\tau_{\boldsymbol{x}}^{\ell}$, using the decomposition

$$\tau_{\boldsymbol{x}} = \tau_{\boldsymbol{u}_1} + \sum_{j=1}^{N} (\tau_{\omega_{\boldsymbol{u}_1}^{(j+1)}} - \tau_{\omega_{\boldsymbol{u}_1}^{(j)}}). \tag{2.20}$$

Consider, e.g., the bounding diagram with $\omega_{\boldsymbol{u}_1}^{(i)} = \boldsymbol{u}_i$ for all $i = 2,\dots,N$; we denote this diagram by $U(\boldsymbol{x})$ for convenience. Then, by (2.18) and (2.10)–(2.11), the contribution from $\tau_{\boldsymbol{u}_1}^{\ell}$ is bounded as

$$\sum_{\substack{\boldsymbol{u}_1,\dots,\boldsymbol{u}_{N+1}\\ \boldsymbol{v}_1,\dots,\boldsymbol{v}_{N+1}\\ (\boldsymbol{u}_{N+1}=\boldsymbol{v}_{N+1})}} \tau_{\boldsymbol{u}_1}^{\ell} \varphi_p(\boldsymbol{u}_1)\, \varphi_p^{(m)}(\boldsymbol{v}_1) \prod_{i=1}^{N} \varphi_p(\boldsymbol{u}_i,\boldsymbol{v}_i)\, \xi_p^{||}(\boldsymbol{u}_i,\boldsymbol{v}_i;\boldsymbol{u}_{i+1},\boldsymbol{v}_{i+1}) m^{\tau_{\boldsymbol{v}_{i+1}}-\tau_{\boldsymbol{v}_i}} \le (T_p^{(m)})^N \times \begin{cases} T_p^{(m)} & (\ell=1), \\ S_p^{(m)} & (\ell=2), \end{cases} \tag{2.21}$$

and, by (2.19) and (2.10)–(2.11) and using (2.15), the contribution from each $(\tau_{\boldsymbol{u}_{j+1}}-\tau_{\boldsymbol{u}_j})^\ell$ is bounded as

$$\begin{aligned}
&\sum_{\substack{\boldsymbol{u}_1,\dots,\boldsymbol{u}_{N+1}\\ \boldsymbol{v}_1,\dots,\boldsymbol{v}_{N+1}\\ (\boldsymbol{u}_{N+1}=\boldsymbol{v}_{N+1})}} \varphi_p(\boldsymbol{u}_1)\,\varphi_p(\boldsymbol{u}_1;\boldsymbol{v}_1)\,\varphi_p^{(m)}(\boldsymbol{v}_1)\bigg(\prod_{i=1}^{j-1}\xi_p^{||}(\boldsymbol{u}_i,\boldsymbol{v}_i;\boldsymbol{u}_{i+1},\boldsymbol{v}_{i+1})m^{\tau_{\boldsymbol{v}_{i+1}}-\tau_{\boldsymbol{v}_i}}\varphi_p(\boldsymbol{u}_{i+1};\boldsymbol{v}_{i+1})\bigg)\\
&\qquad\qquad\times(\tau_{\boldsymbol{u}_{j+1}}-\tau_{\boldsymbol{u}_j})^\ell\,\xi_p^{||}(\boldsymbol{u}_j,\boldsymbol{v}_j;\boldsymbol{u}_{j+1},\boldsymbol{v}_{j+1})m^{\tau_{\boldsymbol{v}_{j+1}}-\tau_{\boldsymbol{v}_j}}\\
&\qquad\qquad\times\bigg(\prod_{i=j+1}^{N}\varphi_p(\boldsymbol{u}_i;\boldsymbol{v}_i)\,\xi_p^{||}(\boldsymbol{u}_i,\boldsymbol{v}_i;\boldsymbol{u}_{i+1},\boldsymbol{v}_{i+1})m^{\tau_{\boldsymbol{v}_{i+1}}-\tau_{\boldsymbol{v}_i}}\bigg)\\
&\le(1+2T_p^{(m)})(T_p^{(m)})^{N-1}\times\begin{cases}T_p^{(m)} & (\ell=1),\\ S_p^{(m)} & (\ell=2).\end{cases}
\end{aligned}\tag{2.22}$$

Therefore, for $\ell=1$,

$$\sum_{\boldsymbol{x}}\tau_{\boldsymbol{x}}U(\boldsymbol{x})m^{\tau_{\boldsymbol{x}}}\le N(1+2T_p^{(m)})(T_p^{(m)})^N+(T_p^{(m)})^{N+1}\le(N+1)(1+2T_p^{(m)})(T_p^{(m)})^N.\tag{2.23}$$

The other $2^{N-1}-1$ bounding diagrams obey the same bound. This completes the proof of (2.6) for $\ell\le1$.

The cross terms for $\ell=2$ can also be bounded similarly. For example, the contribution from $(\tau_{\boldsymbol{u}_{j'+1}}-\tau_{\boldsymbol{u}_{j'}})(\tau_{\boldsymbol{u}_{j+1}}-\tau_{\boldsymbol{u}_j})$ with $j'<j$ is bounded, by using (2.19) (cf., (2.22)), by

$$\begin{aligned}
&(T_p^{(m)})^{N-j'}\sum_{\substack{\boldsymbol{u}_1,\dots,\boldsymbol{u}_{j'+1}\\ \boldsymbol{v}_1,\dots,\boldsymbol{v}_{j'+1}}}\varphi_p(\boldsymbol{u}_1)\,\varphi_p(\boldsymbol{u}_1;\boldsymbol{v}_1)\,\varphi_p^{(m)}(\boldsymbol{v}_1)\prod_{i=1}^{j'-1}\xi_p^{||}(\boldsymbol{u}_i,\boldsymbol{v}_i;\boldsymbol{u}_{i+1},\boldsymbol{v}_{i+1})m^{\tau_{\boldsymbol{v}_{i+1}}-\tau_{\boldsymbol{v}_i}}\varphi_p(\boldsymbol{u}_{i+1};\boldsymbol{v}_{i+1})\\
&\qquad\qquad\times(\tau_{\boldsymbol{u}_{j'+1}}-\tau_{\boldsymbol{u}_{j'}})\,\xi_p^{||}(\boldsymbol{u}_{j'},\boldsymbol{v}_{j'};\boldsymbol{u}_{j'+1},\boldsymbol{v}_{j'+1})m^{\tau_{\boldsymbol{v}_{j'+1}}-\tau_{\boldsymbol{v}_{j'}}}\varphi_p(\boldsymbol{u}_{j'+1};\boldsymbol{v}_{j'+1})\\
&\le(1+2T_p^{(m)})(T_p^{(m)})^{N-1}S_p^{(m)}.
\end{aligned}\tag{2.24}$$

There are $N(N-1)-1$ more cross terms that obey the same bound. There are $2N$ cross terms remaining, each of which is bounded by $(T_p^{(m)})^N S_p^{(m)}$. Therefore,

$$\begin{aligned}
\sum_{\boldsymbol{x}}\tau_{\boldsymbol{x}}^2U(\boldsymbol{x})m^{\tau_{\boldsymbol{x}}}&\le N^2(1+2T_p^{(m)})(T_p^{(m)})^{N-1}S_p^{(m)}+(2N+1)(T_p^{(m)})^N S_p^{(m)}\\
&\le(N+1)^2(1+2T_p^{(m)})(T_p^{(m)})^{N-1}S_p^{(m)}.
\end{aligned}\tag{2.25}$$

The other $2^{N-1}-1$ bounding diagrams than $U(\boldsymbol{x})$ obey the same bound. This completes the proof of (2.6).

Finally we prove (2.7) for $N\ge1$. If $m<1$, then we multiply $\omega_{\boldsymbol{v}_1}$ by $m^{\tau_{\boldsymbol{x}}}$ as before, and multiply $\omega_{\boldsymbol{u}_1}$ by $1-\cos(k\cdot\sigma_{\boldsymbol{x}})$ and use the decomposition (cf., [4, (4.50)])

$$1-\cos(k\cdot\sigma_{\boldsymbol{x}})\le(2N+3)\bigg(1-\cos(k\cdot\sigma_{\boldsymbol{u}_1})+\sum_{j=1}^N\Big(1-\cos\big(k\cdot(\sigma_{\omega_{\boldsymbol{u}_1}^{(j+1)}}-\sigma_{\omega_{\boldsymbol{u}_1}^{(j)}})\big)\Big)\bigg).\tag{2.26}$$

For example, consider the bounding diagram $U(\boldsymbol{x})$ again, where $\omega^{(i)}_{\boldsymbol{u}_1} = \boldsymbol{u}_i$ for all $i = 2, \ldots, N$. Similarly to (2.21)–(2.22), we have

$$\sum_{\substack{\boldsymbol{u}_1,\ldots,\boldsymbol{u}_{N+1}\\ \boldsymbol{v}_1,\ldots,\boldsymbol{v}_{N+1}\\ (\boldsymbol{u}_{N+1}=\boldsymbol{v}_{N+1})}} \big(1-\cos(k\cdot\sigma_{\boldsymbol{u}_1})\big)\varphi_p(\boldsymbol{u}_1)\,\varphi_p^{(m)}(\boldsymbol{v}_1)\prod_{i=1}^{N}\varphi_p(\boldsymbol{u}_i,\boldsymbol{v}_i)\,\xi_p^{||}(\boldsymbol{u}_i,\boldsymbol{v}_i;\boldsymbol{u}_{i+1},\boldsymbol{v}_{i+1})m^{\tau_{\boldsymbol{v}_{i+1}}-\tau_{\boldsymbol{v}_i}}$$
$$\le W_p^{(m)}(k)(T_p^{(m)})^N, \tag{2.27}$$

and

$$\sum_{\substack{\boldsymbol{u}_1,\ldots,\boldsymbol{u}_{N+1}\\ \boldsymbol{v}_1,\ldots,\boldsymbol{v}_{N+1}\\ (\boldsymbol{u}_{N+1}=\boldsymbol{v}_{N+1})}} \varphi_p(\boldsymbol{u}_1)\,\varphi_p(\boldsymbol{u}_1;\boldsymbol{v}_1)\,\varphi_p^{(m)}(\boldsymbol{v}_1)\bigg(\prod_{i=1}^{j-1}\xi_p^{||}(\boldsymbol{u}_i,\boldsymbol{v}_i;\boldsymbol{u}_{i+1},\boldsymbol{v}_{i+1})m^{\tau_{\boldsymbol{v}_{i+1}}-\tau_{\boldsymbol{v}_i}}\varphi_p(\boldsymbol{u}_{i+1};\boldsymbol{v}_{i+1})\bigg)$$
$$\times\Big(1-\cos\big(k\cdot(\sigma_{\boldsymbol{u}_{j+1}}-\sigma_{\boldsymbol{u}_j})\big)\Big)\xi_p^{||}(\boldsymbol{u}_j,\boldsymbol{v}_j;\boldsymbol{u}_{j+1},\boldsymbol{v}_{j+1})m^{\tau_{\boldsymbol{v}_{j+1}}-\tau_{\boldsymbol{v}_j}}$$
$$\times\bigg(\prod_{i=j+1}^{N}\varphi_p(\boldsymbol{u}_i;\boldsymbol{v}_i)\,\xi_p^{||}(\boldsymbol{u}_i,\boldsymbol{v}_i;\boldsymbol{u}_{i+1},\boldsymbol{v}_{i+1})m^{\tau_{\boldsymbol{v}_{i+1}}-\tau_{\boldsymbol{v}_i}}\bigg)$$
$$\le (1+2T_p^{(m)})(T_p^{(m)})^{N-1}W_p^{(m)}(k). \tag{2.28}$$

Therefore,

$$\begin{aligned}\sum_{\boldsymbol{x}}(1-\cos(k\cdot\sigma_{\boldsymbol{x}}))U(\boldsymbol{x})m^{\tau_{\boldsymbol{x}}} &\le (2N+3)\Big((T_p^{(m)})^N W_p^{(m)}(k)+N(1+2T_p^{(m)})(T_p^{(m)})^{N-1}W_p^{(m)}(k)\Big)\\ &\le 3(N+1)^2(1+2T_p^{(m)})(T_p^{(m)})^{N-1}W_p^{(m)}(k).\end{aligned} \tag{2.29}$$

The other $2^{N-1}-1$ bounding diagrams than $U(\boldsymbol{x})$ obey the same bound.

If $m \ge 1$, then we multiply $\omega_{\boldsymbol{u}_1}$ by $(1-\cos(k\cdot\sigma_{\boldsymbol{x}}))m^{\tau_{\boldsymbol{x}}}$ and use the decomposition (2.26). The rest is the same. This completes the proof of (2.7) for $N \ge 1$. ■

Proof of Lemma 3(ii). First we recall (1.18). Since we have the bound (2.16) on the contribution from $\Xi_p(\boldsymbol{u}_j,\boldsymbol{v}_j;\boldsymbol{u}_{j+1},\boldsymbol{v}_{j+1})$, it thus remains to investigate the contributions from $\Theta_p(\boldsymbol{u}_j,\boldsymbol{v}_j;\boldsymbol{u}_{j+1},\boldsymbol{v}_{j+1})$ and $\Theta'_p(\boldsymbol{u}_j,\boldsymbol{v}_j;\boldsymbol{u}_{j+1},\boldsymbol{v}_{j+1})$. However, since (cf., (2.19))

$$\sum_{\substack{\boldsymbol{u}_1,\ldots,\boldsymbol{u}_{N+1}\\ \boldsymbol{v}_1,\ldots,\boldsymbol{v}_{N+1}\\ (\boldsymbol{u}_{N+1}=\boldsymbol{v}_{N+1})}} \varphi_p(\boldsymbol{u}_1)\,\varphi_p(\boldsymbol{u}_1;\boldsymbol{v}_1)\,\varphi_p(\boldsymbol{v}_1)\bigg(\prod_{i=1}^{j-1}\Xi_p(\boldsymbol{u}_i,\boldsymbol{v}_i;\boldsymbol{u}_{i+1},\boldsymbol{v}_{i+1})\bigg)\bigg(\prod_{i=j+1}^{N}\tilde{\Xi}_p(\boldsymbol{u}_i,\boldsymbol{v}_i;\boldsymbol{u}_{i+1},\boldsymbol{v}_{i+1})\bigg)$$
$$\times\Big(\theta_p^{||}(\boldsymbol{u}_j,\boldsymbol{v}_j;\boldsymbol{u}_{j+1},\boldsymbol{v}_{j+1})+\theta_p^{\times}(\boldsymbol{u}_j,\boldsymbol{v}_j;\boldsymbol{u}_{j+1},\boldsymbol{v}_{j+1})\Big)\le(1+2T_p^{(1)})(2T_p^{(1)})^{N-1}\tilde{T}_p, \tag{2.30}$$

and, for $j < N$,

$$\sum_{\substack{\boldsymbol{u}_1,\ldots,\boldsymbol{u}_{N+1}\\ \boldsymbol{v}_1,\ldots,\boldsymbol{v}_{N+1}\\ (\boldsymbol{u}_{N+1}=\boldsymbol{v}_{N+1})}} \varphi_p(\boldsymbol{u}_1)\,\varphi_p(\boldsymbol{u}_1;\boldsymbol{v}_1)\,\varphi_p(\boldsymbol{v}_1)\bigg(\prod_{i=1}^{j-1}\Xi_p(\boldsymbol{u}_i,\boldsymbol{v}_i;\boldsymbol{u}_{i+1},\boldsymbol{v}_{i+1})\bigg)\bigg(\prod_{i=j+2}^{N}\tilde{\Xi}_p(\boldsymbol{u}_i,\boldsymbol{v}_i;\boldsymbol{u}_{i+1},\boldsymbol{v}_{i+1})\bigg)$$
$$\times\,\Theta'_p(\boldsymbol{u}_j,\boldsymbol{v}_j;\boldsymbol{u}_{j+1},\boldsymbol{v}_{j+1})\Big(\xi_p^{||}(\boldsymbol{u}_{j+1},\boldsymbol{v}_{j+1};\boldsymbol{u}_{j+2},\boldsymbol{v}_{j+2})+\xi_p^{\times}(\boldsymbol{u}_{j+1},\boldsymbol{v}_{j+1};\boldsymbol{u}_{j+2},\boldsymbol{v}_{j+2})\Big)$$
$$\le(1+2T_p^{(1)})(2T_p^{(1)})^{N-2}H_p, \tag{2.31}$$

we obtain

$$\begin{aligned}\sum_{\boldsymbol{x}} \Pi_p^{(N)}(\boldsymbol{x}) &\le (1+2T_p^{(1)})\Big(N(T_p^{(1)}+\tilde{T}_p)(2T_p^{(1)})^{N-1}+(N-1)H_p(2T_p^{(1)})^{N-2}\Big)\\ &\le N(1+2T_p^{(1)})\Big((T_p^{(1)}+\tilde{T}_p)(2T_p^{(1)})^{N-1}+H_p(2T_p^{(1)})^{(N-2)\vee 0}\Big).\end{aligned}\tag{2.32}$$

This completes the proof of Lemma 3(ii). ■

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